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Painlevé test and integrability of nonlinear Klein–Fock–Gordon equations

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Abstract. The applicability of the Painlevé test of the complete integrability of the one-component nonlinear Klein-Fock-Gordon equations in an arbitrary Riemannian space, in the formulation of Weiss *et al* is discussed. Three infinite series of these equations are found in the flat two-dimensional space which possess the Painlevé property and include, as a special case, the Liouville, sine-Gordon, and Dodd-Bullough equations. It is pointed out that the approach of Weiss *et al* to select integrable nonlinear equations is not sufficiently reliable and needs some strengthening.

1. Introduction

Recently Weiss *et al* (1983) proposed a new test of the complete integrability of nonlinear partial differential equations (PDEs) which was based on the generalisation of the Painlevé property known formerly only for ordinary differential equations (ODEs) (Golubev 1941). The connection between ODEs of the Painlevé type and the completely integrable PDEs has been pointed out by Ablowitz and Segur (1977). They suggested that a given PDE should be integrable if it admits an exact reduction to an ODE of the Painlevé type by a suitable symmetry transformation (e.g., by means of the usual similarity variables). In the sequel, we shall use the Painlevé property as was proposed by Weiss *et al* (1983).

The generalisation of the Painlevé property to PDEs reads as follows. Let a solution $\varphi(x_1, \ldots, x_N)$ of a PDE be represented in some domain of C^N as

$$\varphi = \sum_{n=0}^{\infty} \varepsilon_n \omega^{n-k}, \tag{1}$$

where k is a positive integer, ω is a function determining an analytic manifold $\omega(x_1, \ldots, x_N) = 0$ in C^N, along which the poles of φ occur, and ω and $\varepsilon_n(x_1, \ldots, x_N)$ are analytic functions in a neighbourhood of the manifold $\omega = 0$. If the expansion (1) satisfies a given PDE and contains as many arbitrary functions as it should in a general solution of the PDE due to the Cauchy-Kovalevskaya theorem, then this PDE is considered to have the Painlevé property (or is the equation of P type). The Painlevé test for PDEs is formulated in the following way (Weiss *et al* 1983, Weiss 1983): a given PDE is probably completely integrable if it possesses the Painlevé property or can be transformed to a PDE of P type.

In a number of papers (Jimbo *et al* 1982, Weiss *et al* 1983, Weiss 1983, 1984, Chudnovsky and Chudnovsky 1983, Ramani *et al* 1983, Steeb *et al* 1983) the Painlevé property was tested for such PDEs whose integrability is well known. It was found that all the investigated integrable PDEs either possess the Painlevé property in themselves or are transformable to PDEs of P type. What is more, the existence of the Painlevé property for a given PDE turned out to be useful for finding Lax pairs and Bäcklund transformations (however, known previously). These results led to a conjecture that the complete integrability is related to the Painlevé property. Nevertheless, up to now there is no answer to the question of whether one can unambiguously draw a conclusion in favour of the complete integrability of a given PDE proceeding from the presence or absence of the Painlevé property for this PDE.

In this paper we perform the Painlevé analysis of some class of one-component nonlinear Klein-Fock-Gordon PDEs in an arbitrary Riemannian space and select equations reducible to PDEs of P type which should be integrable according to the conjecture of Weiss *et al* (1983). The essential requirement made on the class of equations under consideration is the possibility of transforming them into polynomial PDEs. Section 2 is devoted to such a transformation. In § 3 we carry out the Painlevé analysis of the polynomial equations obtained. Section 4 contains a discussion of the results and some conclusions concerning the Painlevé test as it is formulated by Weiss *et al.*

2. The transformation to the polynomial form

Let us consider a general one-component nonlinear Klein-Fock-Gordon PDE

$$\Box\psi = g(\psi), \tag{2}$$

where $\psi = \psi(x_1, \ldots, x_N)$, $g(\psi)$ is an arbitrary nonlinearity, and a space is regarded to be N-dimensional Riemannian with a metric tensor $g_{\alpha\beta}(x)$. The operator $\Box$ is expressed as $\Box\psi = g^{\alpha\beta}\nabla_\alpha\nabla_\beta\psi = g^{\alpha\beta}\psi_{\alpha\beta} - \Gamma^\alpha\psi_\alpha$. Greek indices at the bottom of scalars stand for derivatives with respect to the relevant coordinates, $\Gamma^\alpha = g^{\mu\nu}\Gamma^\alpha{}_{\mu\nu}$, $\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta})$. The space dimension N and metric tensor $g_{\alpha\beta}(x)$ are not fixed.

Thus, we have the PDE

$$g^{\alpha\beta}\psi_{\alpha\beta} - \Gamma^\alpha\psi_\alpha = g(\psi). \tag{3}$$

If $g(\psi)$ is not a polynomial, equation (3) cannot be the PDE of P type, because the expansion (1) of any solution must contain only finite negative powers of ω. Therefore, it is necessary to transform the PDE (3) to a PDE of the polynomial form which can itself possess the Painlevé property. With this end in view, let us make the substitution of the function $\psi = \psi(\varphi)$, $\varphi = \varphi(x_1, \ldots, x_N)$ with subsequent multiplication of both sides of (3) by a proper function $\Omega(\varphi)$ to obtain the PDE of the polynomial form:

$$A(\varphi)(g^{\alpha\beta}\varphi_{\alpha\beta} - \Gamma^\alpha\varphi_\alpha) + B(\varphi)g^{\alpha\beta}\varphi_\alpha\varphi_\beta = C(\varphi). \tag{4}$$

Here A, B and C are polynomials. In the following we shall restict ourselves to just such PDEs which can be transformed to the polynomial form (4) by the above procedure. With the known polynomials A, B and C in (4), it is easy to reconstruct equation (3) because $A = (\mathrm{d}\psi/\mathrm{d}\varphi)\Omega$, $B = (\mathrm{d}^2\psi/\mathrm{d}\varphi^2)\Omega$, $C = g(\psi(\varphi))\Omega$. Equation (4) allows us already to perform the Painlevé analysis by the substitution of the expansion (1) in (4).

3. The Painlevé analysis

According to Weiss *et al* (1983) we demand any solution φ of the PDE (4) to be represented in the form of (1), that is, in a domain of C^N it must have only poles on the manifold of complex dimension $N-1$ determined by the condition $\omega=0$. Here ω and ε_n are considered to be analytic in a neighbourhood of the manifold $\omega=0$, and the manifold $\omega=0$ itself is assumed to be non-characteristic (in other words, $g^{\alpha\beta}\omega_\alpha\omega_\beta \neq 0$).

Let the powers of the polynomials A, B and C be s, p and t and their coefficients at the highest powers of φ be a_0, b_0 and c_0, respectively. Substituting (1) in (4) and collecting the coefficients with minimal power of ω, we obtain

(1) If $(k-1)a_0+kb_0\neq 0$, then the conditions $\varepsilon_0\neq 0$ and $g^{\alpha\beta}\omega_\alpha\omega_\beta\neq 0$ give $\max[k(s+1)+2,\ k(p+2)+2]=kt$. We put $k(s+1)+2=k(p+2)+2=kt$, without essential lack of generality. Taking into account that s, p and t are non-negative integers and $(k+1)a_0+kb_0\neq 0$, we obtain the two following branches:

(a) $k=2$, the powers of A, B and C are $p+1$, p and $p+3$, respectively; $p=0,1,2,\ldots;$

(b) $k=1$, the powers of A, B and C are $p+1$, p and $p+4$, respectively; $p=0,1,2,\ldots.$

(2) If $a_0(k+1)+b_0k=0$, we get for the same reasons:

(a) $k=1$, the powers of A, B and C are $p+1, p$ and $p+3$, respectively; $p=0,1,2,\ldots;$

(b) $k>1$, the powers of A, B and C are $p+1$, p and $p+2$, respectively; $p=0,1,2,\ldots.$

In what follows we shall consider in detail the case $k=2$ with $3a_0+2b_0\neq 0$, more briefly the case $k=1$ with $2a_0+b_0\neq 0$, as well as one significant example for $k=1$ with $2a_0+b_0=0$.

3.1. *The second-order pole*

Let us consider the case $k=2, 3a_0+2b_0\neq 0$. Here

$$\begin{aligned}
\varphi &= \sum_{n=0}^{\infty}\varepsilon_n\omega^{n-2}, \qquad & A &= \sum_{j=0}^{p+1} a_j\varphi^{p+1-j},\\
B &= \sum_{j=0}^{p} b_j\varphi^{p-j}, \qquad & C &= \sum_{j=0}^{p+3} c_j\varphi^{p+3-j}.
\end{aligned} \tag{5}$$

Substituting (5) in (4) and equating coefficients at ω^l to zero for all l separately, we get the following recursion relations:

$$\begin{aligned}
\sum_j \Bigg(&\sum_{\substack{n_1,\ldots,n_{p-j+2}\\ \Sigma n_i = l+2(p-j)+6}} [a_j(n_1-3)(n_1-2)+b_j(n_1-2)(n_2-2)]\varepsilon_{n_1}\ldots\varepsilon_{n_{p-j+2}}\omega_\alpha\omega_\beta g^{\alpha\beta}\\
&+\sum_{\substack{n_1,\ldots,n_{p-j+2}\\ \Sigma n_i = l+2(p-j)+5}} \{2[a_j(n_1-2)+b_j(n_2-2)]\varepsilon_{n_1\alpha}\omega_\beta g^{\alpha\beta}\varepsilon_{n_2}\ldots\varepsilon_{n_{p-j+2}}\\
&+a_j(n_1-2)\varepsilon_{n_1}\ldots\varepsilon_{n_{p-j+2}}(\omega_{\alpha\beta}g^{\alpha\beta}-\omega_\alpha\Gamma^\alpha)\}\\
&+\sum_{\substack{n_1,\ldots,n_{p-j+2}\\ \Sigma n_i = l+2(p-j)+4}} [a_j(\varepsilon_{n_1\alpha\beta}g^{\alpha\beta}-\varepsilon_{n_1\alpha}\Gamma^\alpha)\varepsilon_{n_2}+b_j\varepsilon_{n_1\alpha}\varepsilon_{n_2\beta}g^{\alpha\beta}]\varepsilon_{n_3}\ldots\varepsilon_{n_{p-j+2}}
\end{aligned}$$

$$+\sum_{\substack{n_1,\ldots,n_{p-j+3}\\ \Sigma n_i = l+2(p-j)+6}} (-c_j)\varepsilon_{n_1}\ldots\varepsilon_{n_{p-j+3}}\Bigg) = 0, \tag{6}$$

where $n_i = 0, 1, 2, \ldots$, $a_j = 0$ for $j > p+1$, $b_j = 0$ for $j > p$ and $c_j = 0$ for $j > p+3$.

The recursion relations (6) make it possible to express ε_m in terms of $\varepsilon_{m-1}, \ldots, \varepsilon_0, a_j, b_j, c_j, \omega$ and their derivatives. However, it may happen that some ε_r, called resonances, are not determined by the recursion relations (6). In this case the compatibility conditions $D_r(\omega) = 0$ arise, where D_r are nonlinear differential operators defined by (6). We shall require the expansion (1) to contain two arbitrary functions, namely, ε_r being the resonance for some number r and ω with the compatibility condition satisfied identically, $D_r(\omega) \equiv 0$ for any ω.

We have from (6) with $l = -2p - 6$:

$$\varepsilon_0 = [(6a_0 + 4b_0)/c_0]\omega_\alpha\omega_\beta g^{\alpha\beta}. \tag{7}$$

It is easy to get from (6) the condition for ε_r to be a resonance:

$$a_0 r^2 - (5a_0 + 4b_0)r - (6a_0 + 4b_0) = 0. \tag{8}$$

It should be noted that $a_0 \neq 0$, because $a_0 = 0$ leads to $r = -1$ and there is no resonance. Therefore, the power of the polynomial A is always equal to $p+1$. The equation (8) has the solutions $r = -1$ and $r = 6 + 4(b_0/a_0)$ which correspond to an arbitrary function ω and to the resonance ε_r, respectively. We note that $r \neq 0$ due to the assumption $3a_0 + 2b_0 \neq 0$. So, the Painlevé analysis of the PDE (4) for the second-order pole gives the infinite series of admissible value for r: $r = 1, 2, 3, \ldots$. We shall consider only the first two ones.

(1) $r = 1$. This means $b_0 = -\frac{5}{4}a_0$. From (6) we obtain the compatibility condition $D_1(\omega) \equiv 0$ where

$$D_1(\omega) = \omega_\alpha\omega_\beta\omega_\gamma I_1^{\alpha\beta\gamma} + \omega_{\alpha\lambda}\omega_\beta\omega_\gamma I_2^{\alpha\lambda\beta\gamma} \tag{9}$$

and $I_1^{\alpha\beta\gamma} = 2g^{\alpha\beta}\Gamma^\gamma + g^{\alpha\beta}{}_{,\sigma}g^{\sigma\gamma}$, $I_2^{\alpha\lambda\beta\gamma} = 2(g^{\alpha\beta}g^{\lambda\gamma} - g^{\alpha\lambda}g^{\beta\gamma})$. The requirement $D_1(\omega) \equiv 0$ for any ω gives

$$I_1^{(\alpha\beta\gamma)} = 0, \qquad I_2^{(\alpha\lambda)(\beta\gamma)} = 0, \tag{10}$$

where parentheses stand for symmetrisation. Noting that $I_2^{\alpha\lambda\beta\gamma}g_{\alpha\lambda} = 2g^{\beta\gamma}(1 - N)$, we obtain from (10) $N = 1$ as a necessary condition. Hence, the equation (4) is the ODE in this case, which is not of interest for us.

(2) $r = 2$. This means $b_0 = -a_0$. In the same way we get from $D_2(\omega) \equiv 0$:

$$c_1 = (c_0/a_0)(3a_1 + 2b_1) \tag{11}$$

$$\tilde{I}_1^{(\alpha\beta\mu\nu)} = 0, \tag{12}$$

$$\tilde{I}_2^{(\alpha\lambda)(\beta\mu\nu)} = 0, \tag{13}$$

$$\tilde{I}_3^{((\alpha\lambda)(\beta\varkappa))(\mu\nu)} = 0 \tag{14}$$

where

$$\tilde{I}_1^{\alpha\beta\mu\nu} = -g^{\alpha\beta}g^{\mu\nu}{}_{,\sigma\tau}g^{\sigma\tau} + g^{\alpha\beta}{}_{,\sigma}g^{\mu\nu}{}_{,\tau}g^{\sigma\tau} - 2g^{\alpha\beta}g^{\mu\sigma}\Gamma^\nu{}_{,\sigma} + 2g^{\mu\nu}{}_{,\sigma}g^{\sigma\beta}\Gamma^\alpha + g^{\alpha\beta}g^{\mu\nu}{}_{,\sigma}\Gamma^\sigma + 2g^{\alpha\beta}\Gamma^\mu\Gamma^\nu, \tag{15}$$

$$\tilde{I}_2^{\alpha\lambda\beta\mu\nu}=2(g^{\alpha\lambda}{}_{,\sigma}g^{\sigma\beta}g^{\mu\nu}-g^{\alpha\lambda}g^{\beta\sigma}g^{\mu\nu}{}_{,\sigma}-2g^{\alpha\beta}{}_{,\sigma}g^{\sigma\lambda}g^{\mu\nu}+2g^{\alpha\beta}g^{\lambda\sigma}g^{\mu\nu}{}_{,\sigma}$$
$$+2g^{\alpha\beta}g^{\lambda\mu}\Gamma^{\nu}-2g^{\alpha\lambda}g^{\mu\nu}\Gamma^{\beta}), \tag{16}$$

$$\tilde{I}_3^{\alpha\lambda\beta\kappa\mu\nu}=2(-2g^{\alpha\lambda}g^{\beta\mu}g^{\kappa\nu}-g^{\mu\nu}g^{\alpha\beta}g^{\lambda\kappa}+2g^{\alpha\mu}g^{\beta\nu}g^{\kappa\lambda}+g^{\alpha\lambda}g^{\beta\kappa}g^{\mu\nu}). \tag{17}$$

Calculating $\tilde{I}_3^{\alpha\lambda\beta\kappa\mu\nu}g_{\alpha\lambda}g_{\beta\kappa}$, we obtain from (14) and (17)

$$(N^2-3N+2)g^{\mu\nu}=0, \tag{18}$$

that is, $N=1$ (ODE) or $N=2$ (PDE). Let us consider the case $N=2$. By a proper coordinate transformation $(x_1, x_2)\to(x, y)$ the matrix $g^{\mu\nu}$ can be brought in some domain into the form

$$g^{\mu\nu}=\begin{pmatrix} a(x,y) & 0 \\ 0 & 1\end{pmatrix}. \tag{19}$$

When $g^{\mu\nu}$ is taken in the form (19), then (13) and (14) are satisfied identically, while (12) puts one condition on $a(x, y)$:

$$aa_{yy}=\tfrac{3}{2}(a_y)^2. \tag{20}$$

When $N=2$, the curvature tensor $R_{\alpha\beta\mu\nu}$ has the single non-trivial component, R_{1212}. Taking into account (19) and (20) we have

$$R_{\alpha\beta\mu\nu}\equiv 0. \tag{21}$$

Hence, the requirement $D_2(\omega)\equiv 0$ for any ω fixes in this case only the coefficients b_0 and c_1, the space dimension ($N=2$) and the geometry (flat). All the other coefficients of the polynomials A, B, and C and the power p are arbitrary. Therefore, we have the infinite series of the PDEs (4) of P type ($p=0, 1, 2, \ldots$) with the second-order poles and resonance ε_2, as well, as the corresponding infinite set of the PDEs (2) transformable to PDEs of P type in two-dimensional flat space. It is easy to write the first representative of this series of the PDEs (2). When $p=0$, we have in the canonical form ($\xi_1\neq 0$)

$$\psi_{xy}=\xi_1 e^{\psi}+\xi_2 e^{-\psi}+\xi_3 e^{-2\psi}, \qquad \xi_i=\text{constant}. \tag{22}$$

This PDE is reduced to the PDE of P type by the substitution $e^{\psi}=\varphi$ with the subsequent multiplication by φ^2. It gives the Liouville equation at $\xi_2=\xi_3=0$, the sine-Gordon equation at $\xi_3=0$ and the Dodd–Bullough equation at $\xi_2=0$. These three equations are the only PDEs of the type (2) whose integrability is known (Ibragimov 1983). However, if the conjecture of Weiss *et al* is true, the above Painlevé analysis indicates the integrability of both the equation (22) and the whole hierarchy of PDEs with $p>0$. This possibility is discussed in § 4.

3.2. *The first-order pole*

Here we consider briefly the second branch: $k=1$ with $2a_0+b_0\neq 0$. After the manner of the case $k=2$ we obtain the recursion relations which give

$$\varepsilon_0=\left(\frac{2a_0+b_0}{c_0}\omega_\alpha\omega_\beta g^{\alpha\beta}\right)^{1/2} \tag{23}$$

and a condition for the existence of the resonance ε_r: $r=4+2(b_0/a_0)$. It should be kept in mind that $r\neq 0$ due to $2a_0+b_0\neq 0$. We shall consider below only the cases $r=1$ and $r=2$.

(1) $r=1$. The compatibility condition $D_1(\omega)\equiv 0$ for any ω determines b_0 and c_1, but leads to $N=1$ (ODE).

(2) $r=2$. The compatibility condition $D_2(\omega)\equiv 0$ for any ω determines b_0, c_1 and c_2:

$$\begin{aligned} b_0 &= -a_0, \\ c_1 &= (c_0/a_0)(4a_1+3b_1), \\ c_2 &= (2c_0/a_0^2)(a_1+b_1)(3a_1+2b_1)+(c_0/a_0)(2a_2+b_2) \end{aligned} \tag{24}$$

and leads to flat two-dimensional space. Other coefficients of the polynomials A, B and C and the power p are not determined. We have again the infinite set of the P type equations (4) and the corresponding infinite set of the PDEs (2) transformable to the PDEs (4) of P type. The first representative of this infinite series of the PDEs (2) is found to be ($\tilde{\xi}_1 \neq 0$)

$$\psi_{xy} = \tilde{\xi}_1\, e^{2\psi} + \tilde{\xi}_2\, e^{-\psi} + \tilde{\xi}_3\, e^{-2\psi}, \qquad \tilde{\xi}_i = \text{constant}, \tag{25}$$

and the substitution $e^{\psi}=\varphi$ with the subsequent multiplication by φ^2 transforms it to the PDE possessing the Painlevé property. The PDE (25) also contains all three known integrable equations of the form (2).

3.3. *The resonance ε_0*

In the preceding consideration when $(k+1)a_0+kb_0\neq 0$ we have obtained that ε_0 could not be the resonance. Let us now take into account the case $(k+1)a_0+kb_0=0$ when ε_0 is the resonance, as is easy to make sure. Here we shall restrict ourselves by the following essential example ($k=1$):

$$\begin{aligned} &\varphi = \sum_{n=0}^{\infty} \varepsilon_n \omega^{n-1}, \qquad A=\varphi^{p+1}, \qquad B=-2\varphi^{p}, \\ &C = c_0\varphi^{p+3}+c_1\varphi^{p+2}+\ldots+c_{p+2}\varphi+c_{p+3}. \end{aligned} \tag{26}$$

In this case ε_0 is not determined (hence, the resonance) and no compatibility conditions on ω arise. Therefore, we arrive at a conclusion that equation (4) with a glance to (26) has the Painlevé property. It is natural to ask now, what equations of the form (2) are transformed to the P type equation (4) with the polynomials (26) and in what way. From the polynomials A and B (26) we obtain $\psi(\varphi)=\mu\varphi^{-1}+\nu$, μ and ν are constants. We put $\mu=1$ and $\nu=0$. Then $\psi(\varphi)=\varphi^{-1}$, $\varphi(\psi)=\psi^{-1}$ and $\Omega(\varphi)=-\varphi^{p+3}$. Since $C(\varphi)=g(\psi(\varphi))\Omega(\varphi)$, we find $g(\psi)=-(c_0+c_1\psi+\ldots+c_{p+3}\psi^{p+3})$. In this case the Painlevé property does not fix the space dimension N, its metric $g_{\alpha\beta}(x)$, the power p and the coefficients c_j. Therefore, the equation

$$\Box\psi = P_n(\psi) \tag{27}$$

where P_n is a polynomial of the power $n\geqslant 3$ is reduced by the substitution $\psi(\varphi)=\varphi^{-1}$ and multiplication by $\Omega(\varphi)=-\varphi^{n}$ to the PDE possessing the Painlevé property, in arbitrary Riemannian space. In fact, the expansion (1) contains two arbitrary function ω and ε_0 with $k=1$. Hence, the application of the Painlevé test of Weiss *et al* to the equation (27) forces us to conclude that (27) should be completely integrable, but it is obviously not the case.

4. Discussion

We have performed the Painlevé analysis of the one-component nonlinear Klein-Fock-Gordon PDEs in an arbitrary Riemannian space with an arbitrary nonlinearity. The special procedure of the transformation of the PDEs in question to PDEs of the polynomial form was employed and some simple cases were studied in every series of admissible values of the polynomial power. Three infinite series of the PDEs (4) possessing the Painlevé property have been selected in the framework of this approach. If the conjecture of Weiss *et al* were valid, then all the obtained equations (4) of P type and the corresponding nonlinear PDEs (2) should be integrable. The result of § 3.3 is especially paradoxical: the PDE (2) in Riemannian space of arbitrary dimension and with an arbitrary polynomial nonlinearity is integrable. Even if we exclude the case when ε_0 is the resonance and the compatibility condition on ω does not occur at all, the integrability of two infinite series of the PDEs possessing the Painlevé property remains problematic. In particular, every one of the two simplest representatives of these series (22) and (25) already contains, as the particular case, all three nonlinear PDEs of the form (2) which have the infinite Lie-Bäcklund transformation group (Ibragimov 1983) which seems to be related to the integrability.

Hence, the above consideration forces us to have doubts about universality of the Painlevé criterion in the formulation of Weiss *et al.* Recently Ward (1984) proposed another version of the Painlevé property: if S is an analytic non-characteristic complex hypersurface in C^N, then every solution of the PDE which is analytic on $C^N\backslash S$, is meromorphic on C^N. He succeeded in establishing the Painlevé property for the self-dual Yang-Mills equations, without the series expansion (1). It is of interest to discuss the Painlevé property of Weiss *et al* from the viewpoint of Ward's test.

(1) As it is pointed out by Ward, some essential singularities may occur in the solution of the PDE during the summation of the expansion (1). Even for ODEs it is often a complicated problem to show that the expansion (1) gives only poles (Golubev 1941). It is significant in this connection that just a truncation of the expansion (1) (Weiss *et al* 1983) leads to obtaining some remarkable properties of nonlinear equations such as Lax pairs and Bäcklund transformations.

(2) The Painlevé analysis of PDEs by means of the expansion (1) is similar to that for ODEs. Nevertheless, there are essential differences between them. In the case of ODEs the coefficients ε_n in (1) are constants and it is required that the recursion relations do not determine q constants where q is the ODE order. Then the expansion (1) can be formally treated as a general solution. In the case of PDEs, ε_n are functions of N variables. The consideration of (1) as the general solution of the PDE is usually motivated by the Cauchy-Kovalevskaya theorem which states that the general solution of the PDE (2) must contain two arbitrary functions of $N-1$ variables: ω and ε_r. This additional freedom is excessive and should disappear in consequence of summing the expansion (1). However, it is impossible to guarantee *a priori* that the needed arbitrariness remains intact in this process and (1) be as before the general solution. Other solutions with essential singularities may occur in this case.

(3) According to the Cauchy-Kovalevskaya theorem, for the expansion (1) to be the general solution of the PDE (4) it is necessary to introduce in (1) two arbitrary functions of $N-1$ variables. In principle, this happens already in the case of no resonances at all (ω is an arbitrary function of N variables) or with one resonance (it could even be fixed) and the compatibility condition for ω being an equation of the order $q \geqslant 2$, rather than an identity (its solution ω will contain q arbitrary functions

of $N-1$ variables). In such cases almost all PDEs possess the Painlevé property. Therefore, both the 'correct number of resonances' in the expansion (1) and the requirement for compatibility conditions to be satisfied identically are not derived from the Cauchy-Kovalevskaya theorem and represent additional postulates.

In accordance with the above arguments concerning the possibility to represent the general meromorphic solution of the nonlinear PDE in the form (1) we conclude that the application of the Painlevé property in the formulation of Weiss *et al* to estabish the complete integrability of the PDE ought to be carried out with care. This circumstance demands a stronger version of the Painlevé test to be formulated.

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